1 Chapter 1

1.1 Rings, Ideals, Radicals

1. (AM Ex. 1). Let $x \in A$ nilpotent and $u \in A$ unit.

   (a) Show that $x + u$ is a unit.

   (b) Let I ideal of A contained in the nilradical and prove that if $u \in A$ is such that $\overline{u} \in A/I$ is a unit, then u itself is a unit in A.

2. (AM Ex. 2). Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

   (a) f is a unit of $A[x]$ if and only if all the coefficients but the constant term are nilpotents of A and the constant term is a unit of A.

   (b) f is nilpotent if and only if all the coefficients are nilpotent.

   (c) f is a zero-divisor if and only if f is annihilated by a nonzero element of A.

   (d) f is *primitive* if its coefficients generate the unit ideal. Prove that a product is primitive if and only if its coefficients are primitive.
   (Note: if the ring A is a unique factorization domain, the word "primitive" has a slightly different meaning: in that context it means the coefficients do not have a nonunit common factor. The two meanings coincide if the ring is a principal ideal domain.)

3. (AM Ex. 4). Show that in $A[x]$, the Jacobson radical and nilradical are equal.

4. (AM Ex. 6). A ring A has the property that every ideal not in the nilradical contains a nonzero idempotent (i.e. an element x such that $x^2 = x$). Prove that the nilradical and Jacobson radical of A coincide.

5. (AM Ex. 7). Let A be a ring.

   (a) Show that if all $x \in A$ satisfy $x^n = x$ for some $n > 1$ (depending on x) then every prime ideal of A is maximal.

   (b) Is the converse true? Prove or give counterexample.

6. (AM Ex. 8). Let A be a nonzero ring. Show that the set of all prime ideals has elements that are minimal with respect to inclusion.

7. (AM Ex. 10). Let A be a ring, $\mathfrak{N}$ its nilradical. Show the following are equivalent: (i) A has just one prime ideal; (ii) every element of A is either a unit or nilpotent; (iii) $A/\mathfrak{N}$ is a field.

8. (AM Ex. 11). A ring A is *boolean* if $\forall x \in A$, $x^2 = x$. In a boolean ring, show that
   (a) $2x = 0$.
   (b) Every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p} = \mathbb{F}_2$.
   (c) Every finitely generated ideal in A is principal.
9. (AM Ex. 12). Prove that a local ring contains no idempotent $\neq 0, 1$.
10. Let A be a ring with I, J ideals. Consider $\mathfrak{F}$ =$\{P$ prime ideals $\mid I \subseteq P, J \nsubseteq P\}$. Prove that
$$(\sqrt{I} : J) = \bigcap_{P \in \mathfrak{F}} P$$

1.2 Prime Spectrum

This and the next section set up fundamental tools of algebraic geometry. We gain insight into the geometric objects under study (curves, surfaces, etc.) by looking at the ring of polynomial functions on those objects. We also reverse the process and start with a ring and construct an underlying geometric object of which it can be seen as the "ring of functions." This underlying geometric object is called its *prime spectrum*. The following exercises define the prime spectrum. See the comments below on exercise 16c, and also exercises 26-28, for more context. Also, Exercises 23-24 in Chapter 3 are aimed at fleshing out the way in which it makes sense to think about the ring elements as "functions" on the prime spectrum.

1. (AM Ex. 15). Let A be a ring and let $X = \operatorname{Spec} A$ be the set of prime ideals of A. For arbitrary $E \subseteq A$, define $V(E)$ to be the set of all prime ideals containing E. Check that
   (a) If $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.
   (b) $V(0) = X$ and $V(1) = \emptyset$.
   (c) If $(E_i)_{i \in I}$ is a family of subsets of A, then
$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i)$$
   (d) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

   These results show that sets of the form $V(E)$ are closed under arbitrary intersection and finite union and contain $X, \emptyset$; thus they obey the axioms for the closed sets of a topology; it is called the Zariski topology on $X = \operatorname{Spec} A$.

2. (AM Ex. 16). Describe $\operatorname{Spec} A$ for $A =$

   (a) $\mathbb{Z}$.
   (b) $\mathbb{R}$.
   (c) $\mathbb{C}[x]$.
   Note: The zero ideal is included for technical reasons we will get into later; we think of it as representing a “generic point" of the complex plane. The elements of $\mathbb{C}[x]$ are naturally interpreted as functions on $\mathbb{C}$; thus in this case, the elements of the ring are naturally thought of as functions on the prime spectrum of the ring. We will take this as a cue and, even where it is a less natural interpretation, we will tend to think of elements of a ring as “functions" on the ring's prime spectrum.
   (d) $\mathbb{R}[x]$.
   (e) $\mathbb{Z}[x]$.
   Note: It might be necessary in proving the classification below of the prime ideals of $\mathbb{Z}[x]$, to refer to Gauss' Lemma, which is not discussed in Atiyah-MacDonald. It is covered in any standard introductory text on abstract algebra such as Artin, *Algebra*.

3. (AM Ex. 17) If $f \in A$, let X_f be the complement of $V(f)$ in $X = \operatorname{Spec} A$. (In the geometric picture based on $A = k[x_1, \ldots, x_n]$, X_f is the complement of a hypersurface...) Prove the following:

   (a) The X_f form a basis for the Zariski topology.
   (b) $X_f \cap X_g = X_{fg}$.
   (c) $X_f = \emptyset \Leftrightarrow f$ is nilpotent.
   (d) $X_f = X \Leftrightarrow f$ is a unit.
   (e) $X_f = X_g$ if and only if (f) and (g) have the same radical.
   (f) X is quasicompact.
   Note: The word “compact", meaning, as usual, that every open cover has a finite subcover, tends to be replaced with the word “quasicompact", because this property is possessed by most of the spaces under study, even if they are not what we are used to thinking of as compact. Fore example, $\operatorname{Spec} \mathbb{C}[x]$, the algebraic-geometric model of the topological space $\mathbb{C}$, is quasicompact, even though it is not compact in the Euclidean topology. There are other more advanced concepts that do a better job of substituting for the usual notion of compactness.
   (g) More generally, each X_f is quasicompact.

(h) An open subset of X is quasicompact if and only if it is a finite union of X_f's.

4. (AM Ex. 18) Let $x \in \operatorname{Spec} A$ be a point of $\operatorname{Spec} A$ the topological space, and let $\mathfrak{p}_x$ be the same element of $\operatorname{Spec} A$ except stressing that it is a prime ideal of A.

   (a) Show $\{x\} \subseteq \operatorname{Spec} A$ is closed if and only if $\mathfrak{p}_x$ is maximal.

   (b) Show the closure of $\{x\}$ is $V(\mathfrak{p}_x)$.

   (c) $y \in \overline{\{x\}} \Leftrightarrow \mathfrak{p}_x \subseteq \mathfrak{p}_y$.

   (d) X is a T_0 space, i.e. any two points are separated by an open set containing one and not the other.

5. (AM Ex. 21). Let $\phi : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec} A, Y = \operatorname{Spec} B$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, i.e. a point of X. So ϕ induces a mapping $\phi^* : Y \to X$. (This map is called the *pullback* of ϕ.) Show that

   (a) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and thus that ϕ^* is continuous.

   (b) If $\mathfrak{a}$ is an ideal of A, then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

   (c) If $\mathfrak{b} \lhd B$, then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

   (d) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\ker\phi)$ of X. (In particular, $\operatorname{Spec} A$ and $\operatorname{Spec} A/\mathfrak{N}$ are naturally homeomorphic.)

   (e) If ϕ is injective, then $\phi^*(Y)$ is dense in X. More generally, $\phi^*(Y)$ is dense in $X \Leftrightarrow \ker\phi \subseteq \mathfrak{N}$.

   (f) Let $\psi : B \to C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

   (g) Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be A's field of fractions. Let $B = A/\mathfrak{p} \times K$. Define $\phi : A \to B$ by $\phi(x) = (\bar{x}, x)$, where $\bar{x}$ is the image of x in $A/\mathfrak{p}$. Show that ϕ^* is bijective but not a homeomorphism.

1.3 Affine Varieties

1. (AM Ex. 26). Here Atiyah and MacDonald define MaxSpec (the set of maximal ideals), noting that in general it does not have the nice functorial properties of Spec, because maximal ideals don't always pull back to maximal ideals. But in some cases it is useful because the elements of MaxSpec can be identified with the points of a topological space.

   Let X be a compact hausdorff topological space and let $C(X)$ be the ring of continuous real-valued functions on X. For $x \in X$, let $\mathfrak{m}_x$ be the ideal of functions vanishing at x. It is maximal because it is the kernel of the homomorphism $C(X) \to \mathbb{R}$ that maps $f \mapsto f(x)$, and this homomorphism is surjective with image the field $\mathbb{R}$. So $x \mapsto \mathfrak{m}_x$ is a mapping μ of X into $\tilde{X} = \operatorname{MaxSpec} C(X)$. The problem aims to show μ is a homeomorphism.

(a) Show that μ is surjective: in other words, every maximal ideal of $C(X)$ has the form $\mathfrak{m}_x$.

(b) By Urysohn's lemma, the continuous functions separate the points of x. Thus show μ is injective.

(c) Let $f \in C(X)$. Let $U_f = \{x \in X : f(x) \neq 0\}$. (I feel Atiyah and MacDonald could have called this X_f to stress the connection with the notation in Exercises 17 and 21.) Let $\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} : f \notin \mathfrak{m}\}$. Show that $\mu(U_f) = \tilde{U}_f$. Show that the open sets U_f, resp. $\tilde{U}_f$, form a basis for the topology of X, resp. $\tilde{X}$, and thus μ is a homeomorphism. (This is a motivating example for algebraic geometry because it shows that the geometric structure of X can be recovered from the ring $C(X)$.)

Thus X can be reconstructed as a topological space from $C(X)$.

2. (AM Ex. 27). Let k be an algebraically closed field and let

$$f_\alpha(t_1, \ldots, t_n) = 0$$

be a set of polynomial equations (indexed by α) in n variables, with coefficients in k. The set X of all points $x = (x_1, \ldots, x_n) \in k^n$ which satisfy these equations is an *affine algebraic variety*.

Consider the set of all polynomials $g \in k[t_1, \ldots, t_n]$ with the property that $g(x) = 0$ for all $x \in X$. Check that this set is an ideal $I(X)$ in the polynomial ring. It is called the *ideal of the variety* X. The quotient ring

$$k[X] = k[t_1, \ldots, t_n]/I(X)$$

is the ring of polynomial functions on X, because two polynomials g, h define the same function on X if and only if $g - h$ vanishes at every point of X, that is, if and only if $g - h \in I(X)$.

Let ξ_i be the image of t_i in $k[X]$. The ξ_i (for $1 \leq i \leq n$) are the *coordinate functions* on X: if $x \in X$, then $\xi_i(x)$ is the ith coordinate of x. $k[X]$ is generated as a k-algebra by the coordinate functions, so is called the *coordinate ring* (or affine algebra) of X.

As in Exercise 26, for each $x \in X$ let $\mathfrak{m}_x$ be the ideal of all $f \in k[X]$ such that $f(x) = 0$; check that it is a maximal ideal of $k[X]$. Hence, if $\tilde{X} = \operatorname{MaxSpec}(k[X])$, we have defined a mapping $\mu : X \to \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

It is easy to show that μ is injective: if $x \neq y$, we must have $x_i \neq y_i$ for some i $(1 \leq i \leq n)$, and hence $\xi_i - x_i$ is in $\mathfrak{m}_x$ but not in $\mathfrak{m}_y$, so that $\mathfrak{m}_x \neq \mathfrak{m}_y$. What is less obvious (but still true) is that μ is *surjective*. This is one form of Hilbert's Nullstellensatz (see chapter 7).

Note: this discussion shows that, as in Exercise 26, the MaxSpec of the ring $k[X]$ is in bijection with the points of X. If we take a subset of X to be closed if it is defined by the vanishing of some polynomials, we get a topology on X called the Zariski topology, and this bijection also identifies this topology with the topology of $\operatorname{MaxSpec} k[X]$. Thus again we get a way to go back and forth between a topological space, X, and a ring of functions $k[X]$ on this topological space. The next exercise shows how algebraic maps between two affine varieties X and Y can be turned into corresponding ring homomorphisms between their respective rings of functions. This is the complement (in the concrete situation of affine varieties) of the process described in Exercise 21, which shows (in the more general context of an arbitrary ring) how to take a ring homomorphism and turn it into a continuous map between topological spaces.

3. (AM Ex. 28). Let $f_1, \dots, f_m$ be elements of $k[t_1, \dots, t_n]$. They determine a *polynomial mapping* $\phi : k^n \to k^m$: if $x \in k^n$, the coordinates of $\phi(x)$ are $f_1(x), \dots, f_m(x)$.

   Let X, Y be affine algebraic varieties in k^n, k^m respectively. A mapping $\phi : X \to Y$ is said to be *regular* if ϕ is the restriction to X of a polynomial mapping from k^n to k^m.

   If η is a polynomial function on Y, then $\eta \circ \phi$ is a polynomial function on X. Hence ϕ induces a k-algebra homomorphism $k[Y] \to k[X]$, namely $\eta \mapsto \eta \circ \phi$. Show that in this way we obtain a one-to-one correspondence between regular mappings $X \to Y$ and k-algebra homomorphisms $k[Y] \to k[X]$.